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10/31 愛工大研究会 "Geometry and topology of discrete groups"

Eigenvalue maximization and inflation of maps

## Nash embedding and infrated map

**Thm** (Nash 1956)

Any compact Riem. mfd $(M^n, h)$ admits an isometric embedding $\varphi : M \to \mathbb{R}^N$, $N = \frac{n(3n+11)}{2}$.

The image $\varphi(M)$ can be made arbitrary small if one allows $N$ bigger, but cannot be very large:

$$\mathrm{diam}(\varphi(M)) \leq \mathrm{diam}(N, h).$$

**Question** What is the "largest" isometric embedding?

Fix $d\mu = \cancel{\text{volume element of unit volume}}$ $d\mu_h / \mathrm{Vol}(h)$

$h$ Riem. metric

on $M$.

**Problem 1** Over all $C^\infty$-maps $\varphi : M \to \ell^2$ $\left( = \{(a_n) \subset \mathbb{R}^\infty \mid \sum a_n^2 < \infty\} \right)$ satisfying

$$\varphi^* g_{\ell^2} \leq h, \quad \bar{\varphi} = \int_M \varphi \, d\mu = 0,$$

maximize the variance

$$\mathrm{var}(\varphi) := \int_M \|\varphi\|^2 d\mu.$$

**Def** A map maximizing var, if exists, is called an infrated map.

## Eigenvalue maximization

As a dual of Problem 1, we obtain an e.v. max. problem.

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$g$ Riem. metric on $M$

$\lambda_1(d\mu, g)$ first eigenvalue of Bakry-Émery Laplacian

$$-\Delta_{(d\mu, g)} = -\Delta_g \varphi + g(\nabla f, \nabla \varphi), \quad d\mu_g = e^f d\mu$$

<u>Problem 2</u> Over all Riem. metrics $g$ satisfying

$$\int_M \mathrm{tr}_g h \, d\mu = 1,$$

minimize $1/\lambda_1(d\mu, g)$.

Duality implies

$$\sup_\varphi \mathrm{var}(\varphi) \le \inf_g 1/\lambda_1(d\mu, g)$$

<u>Lemma</u> In the inequality

$$\mathrm{var}(\varphi) \le 1/\lambda_1(d\mu, g),$$

the equality holds iff

$$(\#) \quad \begin{cases} -\Delta(d\mu, g)\, \varphi = \lambda_1(d\mu, g)\, \varphi \\ \varphi^* g_{\ell^2} = h \end{cases}$$

<u>Thm</u> $g$ solution to Problem 2

$\Rightarrow$ $\exists\, \varphi_1, \cdots, \varphi_N$ first eigenfunctions of $-\Delta(d\mu, g)$

s.t.

$$\varphi = (\varphi_1, \cdots, \varphi_N) : M \to \mathbb{R}^N$$

is an isometric immersion w.r.t. $h$.

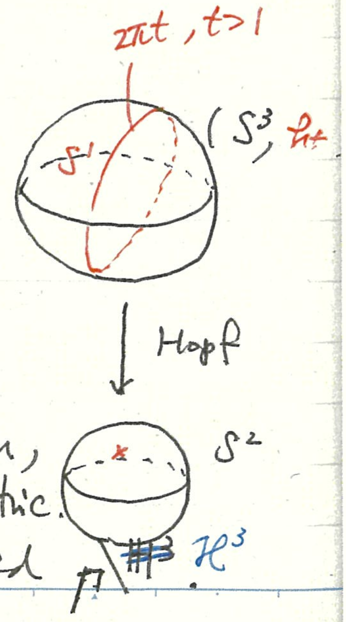
<u>Ex</u> Can solve Problems 1, 2 for

(0) symmetric spaces

(1) flat $T^2$

(2) Berger spheres

If $t > 1$, $\varphi$ is not an isometric immersion, and $g$ is not a Riem. metric, but a CC-metric.




<u>Question</u> Solve Problems 1, 2 for Heisenberg mfd ~~$H^3$~~ $\mathcal{H}^3$.