Continued fraction representations of quadratic irrationals and closed trajectories of modular billiards

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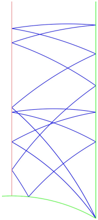
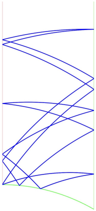




Introduction

- $\Delta = \{z \in \mathbb{C} \mid 0 < z < 1/2,\ |z| > 1\}$: a fundamental domain of $GL(2,\mathbb{Z}) \curvearrowright \mathbb{H}^2$.
- An oriented geodesic γ in $\mathbb{H}^2$

$\rightsquigarrow$ The trajectory of a modular billiard (or billiard) B in $\Delta = \mathbb{H}^2/GL(2,\mathbb{Z})$.

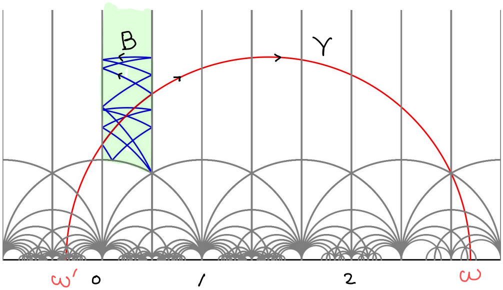
- The billiard B is closed

$\iff$ end points of γ are quadratic irrational ω and its conjugate ω'

$\iff$ ω is (equivalent to) a purely periodic continued fraction $[\overline{a_0, a_1, \ldots a_k}]$

We want to understand the relation "billiard $\leftrightsquigarrow$ period."

There are lot of references based on $SL(2,\mathbb{Z})$, but a few on $GL(2,\mathbb{Z})$.



$\omega = [\overline{3.1.2}]$

Continued Fractions

Definition 1

Continued fraction is a rational number of the form

$$[a_0, a_1, \dots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \, \cfrac{\ddots}{a_{n-1} + \cfrac{1}{a_n}}}} \qquad (a_0 \in \mathbb{Z}, a_1, a_2, \dots, a_n \in \mathbb{N})$$

e.g.

$$[1,2,3,4] = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{4}}} = 1 + \cfrac{1}{2 + \cfrac{1}{\frac{13}{4}}} = 1 + \cfrac{1}{2 + \frac{4}{13}} = 1 + \cfrac{1}{\frac{30}{13}} = 1 + \frac{13}{30} = \frac{43}{30}.$$

Continued Fraction Expansions of Irrational Numbers

- We can see that $\sqrt{2} = [1, 2, 2, 2, \ldots]$ from

$$\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + \frac{1}{1 + \sqrt{2}} = 1 + \frac{1}{2 + (\sqrt{2} - 1)} = \cdots .$$

- What is the number of the form $x = [1, 1, 1, \ldots]$? Since

$$x = [1, 1, 1, \ldots] = 1 + \cfrac{1}{1 + \cfrac{1}{\ddots}} = 1 + \frac{1}{x},$$

we have $x^2 = x + 1$, thus $x = \dfrac{1 + \sqrt{5}}{2}$. This is the golden number.

Theorem 2

The map

$$\mathbb{Z} \times \mathbb{N}^{\mathbb{N}} \longrightarrow \mathbb{R} \setminus \mathbb{Q}, \qquad (a_n)_{n=0}^{\infty} \mapsto [a_0, a_1, a_2, \ldots]$$

is a bijection.

Quadratic Irrationals

Definition 3

An irrational number ω is called quadratic if ω is a root of a quadratic equation $ax^2 + bx + c = 0$ with integer coefficients. The other root ω' of the equation is called the conjugate of ω.

e.g. • $\omega = \sqrt{2} \Rightarrow \omega' = -\sqrt{2}$. • $\omega = \frac{1+\sqrt{5}}{2} \Rightarrow \omega' = \frac{1-\sqrt{5}}{2}$.

Theorem 4 (Lagrange)

Let $\omega = [a_0, a_1, \ldots]$ be an irrational number. Then the followings are equivalent:

- *ω is a quadratic irrational.*
- *$\omega = [a_0, a_1, \ldots]$ is (altimately) periodic, i.e. $\exists k \in \mathbb{Z}_{\geq 0}$, $l \in \mathbb{N}$ s.t.*

$$\omega = [a_0, \ldots, a_k, \overline{a_{k+1}, \ldots, a_{k+l}}].$$

e.g. • $[4, 2, 3, 7, 3, 7, 3, 7, , \ldots] = [4, 2, \overline{3, 7}] = [4, 2, 3, \overline{7, 3}]$.
• $\sqrt{2} = [1, \overline{2}]$. • $\frac{1+\sqrt{5}}{2} = [\overline{1}]$. • $\sqrt{7} = [2, \overline{1, 1, 1, 4}]$. • $[\overline{1, 1, 1, 4}] = \frac{2+\sqrt{7}}{3}$.

Action of $GL(2,\mathbb{Z})$ on $\mathbb{R}\cup\{\infty\}$

$GL(2,\mathbb{Z}) := \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z},\ \det A = \pm 1 \right\}$: unimodular group.

$SL(2,\mathbb{Z}) := \{A \in GL(2,\mathbb{Z}) \mid \det A = 1\}$: modular group.

The action $GL(2,\mathbb{Z}) \curvearrowright \widehat{\mathbb{R}} := \mathbb{R}\cup\{\infty\}$ is defined as:

$$A \cdot x := \frac{a\,x+b}{c\,x+d}, \quad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z}) \text{ and } x \in \widehat{\mathbb{R}}.$$

Note that $A\cdot x = (-A)\cdot x$ and $(AB)\cdot x = A\cdot(B\cdot x)$.

Lemma 5

For every $x \in \mathbb{R}$, $[a_0,a_1,\cdots,a_n,x] = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}\cdot x.$

(Proof) $\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}\cdot x = a_n + \frac{1}{x}$.

$\begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}\cdot x = \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix}\cdot\left(a_n + \frac{1}{x}\right) = a_{n-1} + \cfrac{1}{a_n + \frac{1}{x}}$, and so on.

Equivalent Relation, Periods

Definition 6 (Equivalent reration)

Let ω, η be quadratic irrationals. We define $\omega \sim \eta$ iff $\exists A \in GL(2, \mathbb{Z})$ s.t. $\eta = A \cdot \omega$.

Lemma 7

Every quadratic irrational $\omega = [a_0, \ldots, a_k, \overline{a_{k+1}, \ldots, a_{k+l}}]$ is equivalent to a purely periodic one $\eta = [\overline{a_{k+1}, \ldots, a_{k+l}}]$.

e.g. For $\omega = [4, 2, \overline{1, 2, 3}]$, set $\eta = [\overline{1, 2, 3}]$. Then

$$\omega = [4, 2, \eta] = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \cdot \eta = \begin{pmatrix} 9 & 4 \\ 2 & 1 \end{pmatrix} \cdot \eta$$

Thus $[4, 2, \overline{1, 2, 3}] \sim [\overline{1, 2, 3}] \sim [\overline{2, 3, 1}] \sim [\overline{3, 1, 2}]$.

- When $\omega \sim [\overline{a_0, a_1, \ldots, a_k}]$, we define the period of ω by the sequence $\langle a_0, a_1, \ldots, a_k \rangle$ determined up to cyclic permutations.

e.g. When $\omega = [4, 2, \overline{1, 2, 3}]$, the period of $[\omega]$ is $\langle 1, 2, 3 \rangle = \langle 2, 3, 1 \rangle = \langle 3, 1, 2 \rangle$.

Purely Periodic Quadratic Irrationals

Proposition 8

Let ω be a quadratic irrational. Then the followings are equivalent:

- *ω is purely periodic, i.e. $\omega = [\overline{a_0, a_1, \ldots, a_k}]$.*
- *ω is reduced, i.e. $\omega > 1$ and $-1 < \omega' < 0$.*

e.g. For $\omega = [\overline{1, 2}]$, we have

$$\omega = [1, 2, \omega] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \cdot \omega = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} \cdot \omega = \frac{3\omega + 1}{2\omega + 1}.$$

Thus $\omega(2\omega + 1) = 3\omega + 1 \Leftrightarrow 2\omega^2 - 2\omega - 1 = 0$. Hence $\omega = \frac{1+\sqrt{3}}{2}$, $\omega' = \frac{1-\sqrt{3}}{2}$.

Corollary 9

Let $\omega = [\overline{a_0, a_1, \ldots, a_k}]$ and put $A := \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}$. Then we have $A \cdot \omega = \omega$ and $A \cdot \omega' = \omega'$.

A is a generator of $\mathrm{Stab}(\omega) := \{M \in GL(2, \mathbb{Z}) \mid M \cdot \omega = \omega\}$.

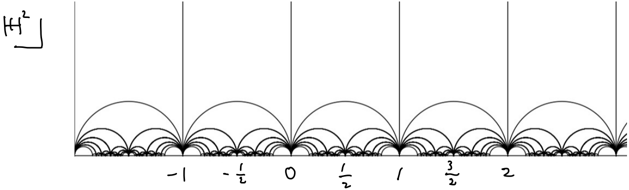
Hyperbolic Geometry (The upper half-plane model)

The upper half-plane model of the hyperbolic plane :
$\mathbb{H}^2 = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ with $ds^2 = \frac{dx^2+dy^2}{y^2}$. Note that $\partial_\infty \mathbb{H}^2 = \widehat{\mathbb{R}}$.

- The geodesics in $\mathbb{H}^2$ are semicircles or straight lines orthogonal to $\mathbb{R}$.
- The action $GL(2,\mathbb{Z}) \curvearrowright \widehat{\mathbb{R}}$ extends to the isometric action $GL(2,\mathbb{Z}) \curvearrowright \mathbb{H}^2$ as

$$A \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \begin{cases} \dfrac{a\,z + b}{c\,z + d} & \text{if } \det A = 1, \\[2ex] \dfrac{a\,\overline{z} + b}{c\,\overline{z} + d} & \text{if } \det A = -1. \end{cases}$$

- Two rational numbers $\frac{p}{q}, \frac{r}{s}$ are called neighbors if $ps - qr = \pm 1$. Joining all neighbors by geodesics in $\mathbb{H}^2$, we obtain the Farrey triangulation $\mathcal{F}$ of $\mathbb{H}^2$. The action of $GL(2,\mathbb{Z})$ on $\mathbb{H}^2$ preserves the Farrey triangulation.

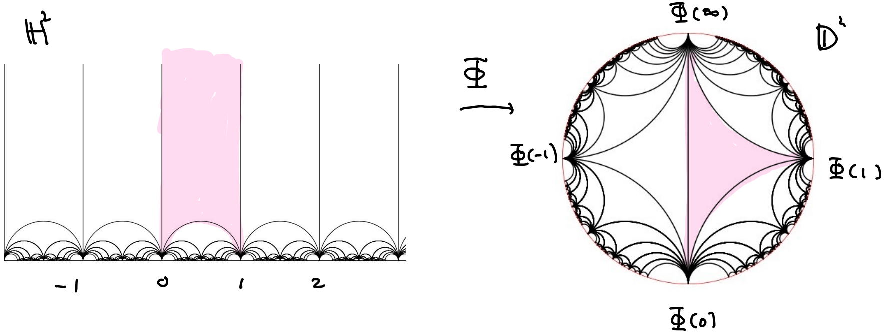


Hyperbolic Geometry (The unit disk model)

The unit disk model of the hyperbolic plane :
$\mathbb{D}^2 = \{z \in \mathbb{C} \mid |z| < 1\}$ with $ds^2 = \frac{|dz|^2}{(1-|z|^2)^2}$. The map

$$\Phi : \mathbb{H}^2 \to \mathbb{D}^2; \quad z \mapsto i\,\frac{z-i}{z+i}$$

is an isometry. From now on, we will identy $\mathbb{D}^2$ with $\mathbb{H}^2$ via Φ.



Generators of $SL(2,\mathbb{Z})$ and $GL(2,\mathbb{Z})$

- Let $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in SL(2,\mathbb{Z})$. Then $SL(2,\mathbb{Z}) = \langle L, R \rangle$.

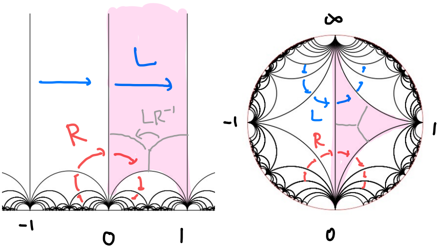
Note that $L \cdot z = z + 1$ and $R \cdot z = z/(z+1)$.

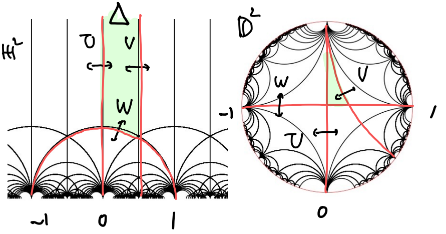
- Let $U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $V = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$, $W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL(2,\mathbb{Z})$.

Then $GL(2,\mathbb{Z}) = \langle U, V, W \rangle = \langle L, W \rangle$.

Note that $U \cdot z = -\bar{z}$, $V \cdot z = -\bar{z} + 1$ and $W \cdot z = 1/\bar{z}$.

The domain $\Delta := \{z \in \mathbb{H}^2 \mid 0 \leq \operatorname{Re} z \leq 1/2,\ |z| > 1\}$ is a fundamental domain of $GL(2,\mathbb{Z}) \curvearrowright \mathbb{H}^2$.



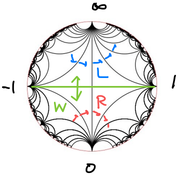


Stabilizers of Purely Periodic Irrationals

Let $\omega = [\overline{a_0, a_1, \ldots, a_n}]$ and put $A := \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$.

Recall that $A \cdot \omega = \omega$, and that A generates $\mathrm{Stab}(\omega)$.

Since $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} = L^a W$, A can be written as a ward in L and W:



$$(\star) \qquad A = L^{a_0} W L^{a_1} W \cdots L^{a_n} W.$$

Furthermore, since $LW = WR$ and $W^2 = I$, A can also be written as a ward in L, R (and W):

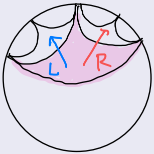
$$(\star\star) \qquad A = \begin{cases} L^{a_0} R^{a_1} \cdots L^{a_{n-1}} R^{a_n} & (n : \text{odd}) \\ L^{a_0} R^{a_1} \cdots L^{a_{n-2}} R^{a_{n-1}} L^{a_n} W & (n : \text{even}) \end{cases}$$

Periods and Cutting Sequences

Proposition 10

Let ω be a purely periodic quadratic irrational and γ a geodesic joining ω' to ω. Then the followings are equivalent:

- $\omega = [\overline{a_0, a_1, \ldots, a_n}]$.
- *The* cutting sequence *of γ across $\mathcal{F}$ is*

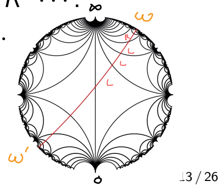


$$\cdots L^{a_{n-1}} R^{a_n} \mid L^{a_0} R^{a_1} L^{a_2} R^{a_3} \cdots$$

where L, R stand for "left" and "right", respectively.

e.g.

- For $\omega = [\overline{3, 1, 2}]$, the cutting sequence is $\cdots R^3 L R^2 \mid L^3 R L^2 R^3 \cdots$.
- For $\omega = [\overline{3, 2}]$, the cutting sequence is $\cdots L^3 R^2 \mid L^3 R^2 L^3 \cdots$



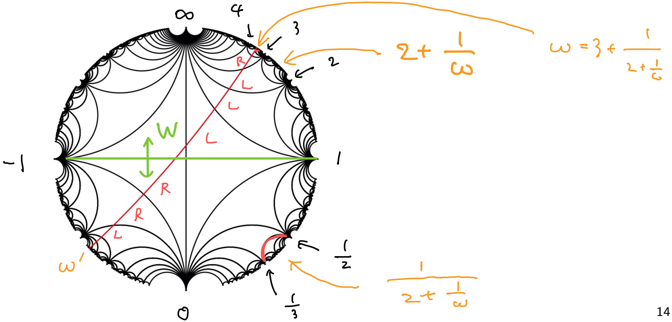
Periods and Cutting Sequences

Proposition 11 (Reprint of Prop. 10)

$\omega = [\overline{a_0, a_1, \ldots, a_n}] \iff$ *The cutting sequence of* γ *across* $\mathcal{F}$ *is* $\cdots L^{a_{n-1}} R^{a_n} \mid L^{a_0} R^{a_1} L^{a_2} R^{a_3} \cdots$ *where* L, R *stand for "left" and "right", respectively.*

For $\omega = [\overline{3, 2}]$, the cutting sequence is $\cdots L^3 R^2 \mid L^3 R^2 L^3 \cdots$.

Note that $\omega = [\overline{3, 2}] = [3, 2, \omega] = L^3 W L^2 W \cdot \omega = 3 + \dfrac{1}{2 + \frac{1}{\omega}}$.



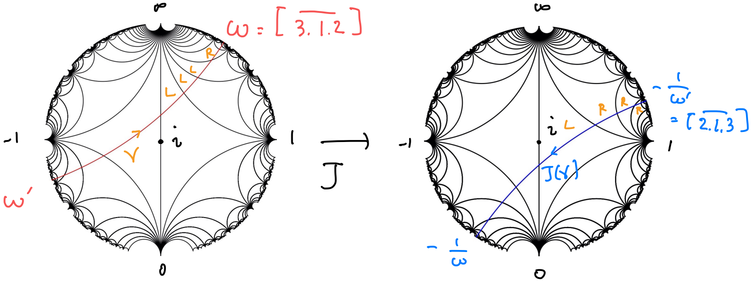
Galois' Theorem

Theorem 12 (Galois)

For $\omega = [\overline{a_0, a_1, \ldots, a_n}]$*, we have* $-\dfrac{1}{\omega'} = [\overline{a_n, \ldots, a_1, a_0}]$.

- $\omega\omega' = -1 \iff (a_0, \ldots, a_n)$ is a palindrome.
- $\omega' \sim [\overline{a_n, \ldots, a_1, a_0}]$.

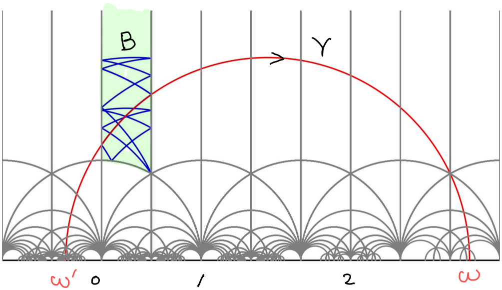
(Proof of Thm) For the geodesic γ from ω' to ω, the cutting sequence is $\cdots L^{a_{n-1}} R^{a_n} | L^{a_0} R^{a_1} \cdots$. If we apply $J \cdot z = -1/z$ on $\mathbb{H}^2$ (where $J = L^{-1}RL^{-1}$), $J(\gamma)$ with its orientation reversed is a geodesic from $-1/\omega$ to $-1/\omega'$ with the cutting sequence $\cdots L^{a_1} R^{a_0} | L^{a_n} R^{a_{n-1}} \cdots$. $\square$

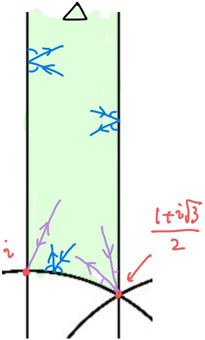


Billiards in Δ

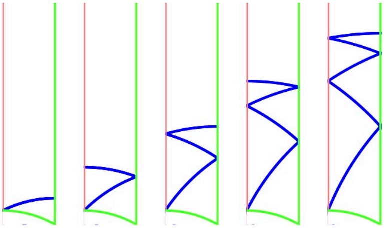
Let $\omega \sim [\overline{a_0, a_1, \ldots, a_n}]$, and γ a geodesic in $\mathbb{H}^2$ joining ω' to ω. Since elements in Stab(ω) map γ to itself, γ descends to a closed trajectory of a billiard B_ω in the orbifold $\Delta = \mathbb{H}^2 / GL(2, \mathbb{Z})$:

$$B_\omega := \left(\bigcup_{A \in GL(2,\mathbb{Z})} A \cdot \gamma \right) \cap \Delta.$$

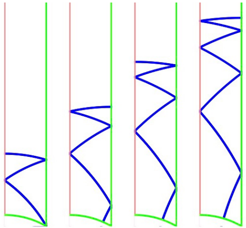




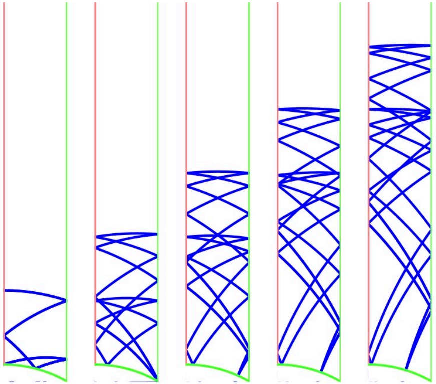
Examples of Billiards



$[\bar{1}]$ $[\bar{2}]$ $[\bar{3}]$ $[\bar{4}]$ $[\bar{5}]$

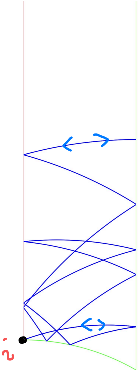
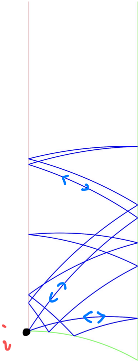
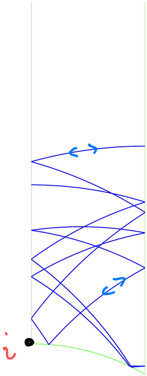
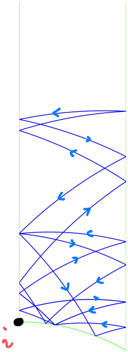


$[\overline{1.2}]$ $[\overline{1.3}]$ $[\overline{1.4}]$ $[\overline{1.5}]$



$[\overline{1.1.2}]$ $[\overline{1.2.3}]$ $[\overline{1.3.4}]$ $[\overline{1.4.5}]$ $[\overline{1.5.6}]$

Billiards and Periods

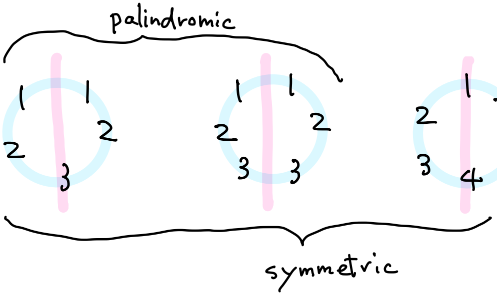
non-orientable			orientable
passes i		doesn't pass i	
once	twice		
e.g. $[\overline{1,2,3,2,1}]$	e.g. $[\overline{1,2,3,3,2,1}]$	e.g. $[\overline{1,2,3,2,1,2}]$	e.g. $[\overline{1,2,3,1}]$
			

Definition 13

Let ω be a quadratic irrational.

- ω is called palindromic if the period of ω has an palindromic expression.
- ω is called symmetric if the period of ω is a union of (at most) two palindromes. (There is a symmetric axis if the period is arranged on a circle.)

$\omega = [4, \overline{1, 1, 3}]$ is palindromic, since its period is $\langle 1, 1, 3\rangle = \langle 1, 3, 1\rangle$.
$\omega = [\overline{1, 3, 1, 4, 5, 4}]$ is symmetric but not palindromic.



Billiards and Periods

Proposition 14

Let ω be a quadratic irrational.

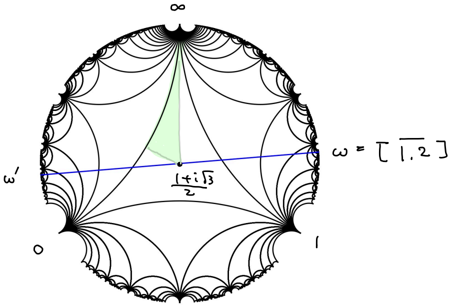
- B_ω *is non-orientable* $\iff$ ω *is symmetric.*
- B_ω *passes* $i \in \partial\Delta$ $\iff$ ω *is palindromic.*
  Furthermore, B_ω *passes* i *twice* $\iff$ *the length of the period of* ω *is even.*

Suppose that $\omega \sim [\overline{a_0, a_1, \ldots, a_n}]$. Then $\omega' \sim [\overline{a_n, \ldots, a_1, a_0}]$.

- B_ω is non-orientable $\iff$ $\omega \sim \omega'$
  $\iff$ $[\overline{a_0, a_1, \ldots, a_n}] \sim [\overline{a_n, \ldots, a_1, a_0}]$
  $\iff$ $(a_0, a_1, \ldots, a_n)$ equals to a cyclic permutation of $(a_n, \ldots, a_1, a_0)$
  $\iff$ ω is symmetric.
- B_ω passes $i \in \partial\Delta$ $\iff$ $\omega \sim \eta$ s.t. $\eta\eta' = -1$ $(\Longrightarrow \omega \sim \omega')$
  $\iff$ $\exists k$ s.t. $[\overline{a_k, a_{k+1}, \ldots, a_{k-1}}] = [\overline{a_{k-1}, \ldots, a_{k+1}, a_k}]$
  $\iff$ ω is palindromic.

Billiards Passing the Corners

- B_ω passes the corner $i \in \partial\Delta \iff \omega \sim \eta$ s.t. $\eta\eta' = -1$
  $\iff \omega$ is equivalent to a solution of $ax^2 + bx - a = 0$.
- B_ω passes the corner $\frac{1+i\sqrt{3}}{2} \in \partial\Delta \iff \omega$ is equivalent to a solution of $ax^2 - 2(a+c)x + c = 0$. For example, $\omega = [\overline{1,2}]$, $[\overline{1,4,1,1}]$ satisfy this condition. Are there any condition on the cycle of ω?



From Billiards to Periods (The Morse Method)

We rabel the sides of $\partial\Delta$ by U, V and W.

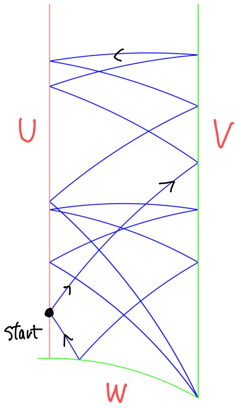
- Let B_ω be the billiard of a quadratic irrational ω.
- Starting from a point in $U \cap B_\omega$ and following B_ω.

Each time the billiard intersects the sides U, V, or W, add that letter to the right.

(When B_ω pass i, we add $WU = UW$, and when B_ω pass $\frac{1+i\sqrt{3}}{2}$, we add $WVW = VWV$.)

- We obtain a ward $g(\omega)$ in U, V and W when one return to the starting point.
- Using $WU = UW$, $VWV = WVW$ and $L = VU$, we can translate $g(\omega)$ into a word $h(\omega) = L^{a_0}WL^{a_1}W \cdots L^{a_n}W$ in L and W.

Then the period of ω is $\langle a_0, a_1, \ldots, a_n \rangle$.



e.g. $g(\omega) = VUVUVUWV\underline{WU}VUV\underline{WU}$

$\rightarrow VUVUVUWV\underline{UW}VUV\underline{UW}$

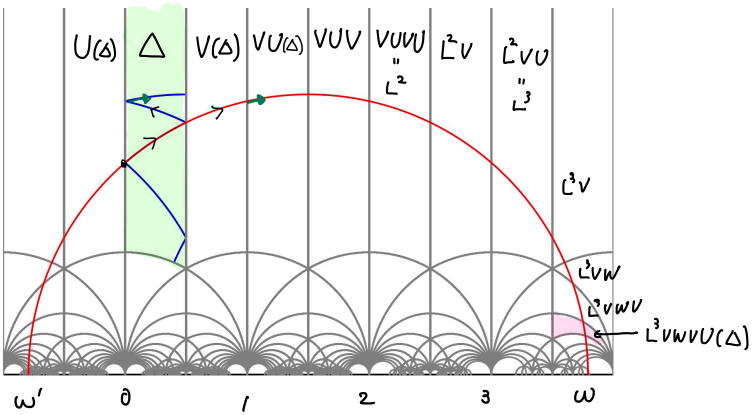
$\rightarrow h(\omega) = LLLWLWLLW \rightarrow \langle 3, 1, 2 \rangle = \langle 1, 2, 3 \rangle$

From Billiards to Periods (The Morse Method)

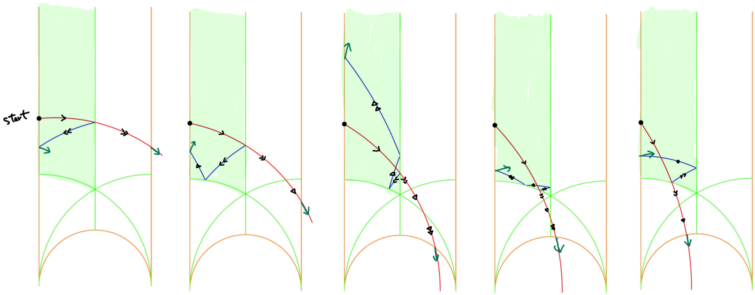
e.g. For $\omega = [\overline{1,3}]$,

$$g(\omega) = VUVUVU\underline{VWV}U = VUVUVU\underline{WVW}U = VUVUVUWVUW$$

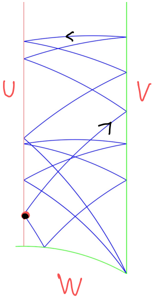
$$h(\omega) = LLLWLW \to \langle 3, 1\rangle = \langle 1, 3\rangle.$$



From Billiards to Periods (The Morse Method)



VU	VWU	VWVU	WVWU	WVU
= L	= VUW	= WVWU	= WVUW	= WL
	= LW	= WVUW	= WLW	
		= WLW		



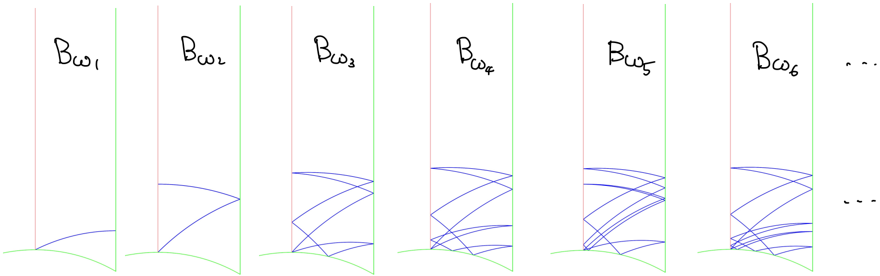
$[\,\overline{3,1.2}\,]$

Markov Spectra

For a modular billiard B, let $\lambda(B)$ be twice of the maximal height of B. The set

$$\mathcal{M} := \{\lambda(B) \mid B \text{ is a modular billiard}\}$$

is called the Markov spectrum. $\mathcal{M} \cap (0,3)$ is a monotone increasing sequence $\{\lambda(B_{\omega_j}) \mid j \in \mathbb{N}\}$ s.t. $\lim_{j\to\infty} B_{\omega_j} = 3$. Here $\omega_1 = [\overline{1}]$, $\omega_2 = [\overline{2}]$, $\omega_3 = [\overline{2,2,1,1}]$, $\omega_4 = [\overline{2,2,1,1,1,1}]$, $\omega_5 = [\overline{2,2,2,2,1,1}]$, $\omega_6 = [\overline{2,2,1,1,1,1,1,1}]\ldots$



cf. N. Andersen and W. Duke, *Markov spectra for modular billiards*, Math. Ann. 373 (2019),1151–1175.

References

1. N. Andersen and W. Duke, *Markov spectra for modular billiards*, Math. Ann. 373 (2019),1151–1175.
2. A. Hatcher, *Topology of numbers*, AMS, 2022.
3. S. Katok, *Continued fractions, hyperbolic geometry and quadratic forms*, Course notes, 2001.
4. S. Katok and I. Ugarcovici, *Symbolic dynamics for the modular surface and beyond*, Bulletin AMS, 44 (2007), 87–132.
5. C. Series, *The geometry of Markoff numbers*, Math. Intelligencer 7 (1985), 20–29.
6. C. Series, *Continued fractions and hyperbolic geometry*, Loughborough LMS summer school, 2015.