On the growth rates of hyperbolic 4-dimensional Coxeter polyhedra

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Geometry and Topology of Discrete Groups

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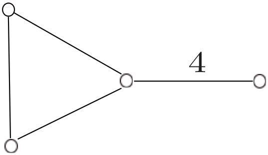
1 Growth rates of hyperbolic Coxeter groups

Coxeter systems

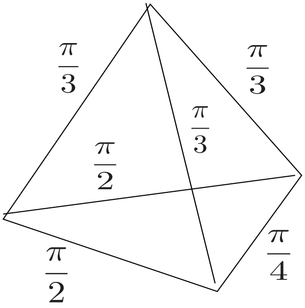
S: a finite set
$M = (m_{r,s}) : |S| \times |S|$ symmetric matrix satisfying
$m_{s,s} = 1, m_{r,s} = m_{s,r} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ (Coxeter matrix)
$W = W(M) = \langle S \mid (rs)^{m_{r,s}} \ (r, s \in S) \rangle$ (Coxeter group)
(W, S) the Coxeter system
The Coxeter diagram of (W, S) is the non-oriented graph whose vertices correspond to S; their vertices r and s are connected by an edge with weight $m_{r,s}$ if $m_{r,s} \geq 3$. We also omit the weight $m_{r,s} = 3$.



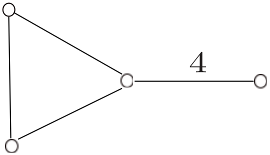
A hyperbolic Coxeter polytope is the intersection of closed half spaces in $\mathbb{H}^n$

$$P = \cap_{i=1}^{k} H_i^- \subset \mathbb{H}^n$$

whose dihedral angles are of the form π/n, $(n \in \mathbb{N}_{\geq 2} \cup \{\infty\})$.
Suppose that S is the set of facets of P. Then P is represented by its Coxeter diagram. A geometric Coxeter group (W, S) is generated by the set of reflections with respect to facets of P.



(a) Coxeter polytope in $\mathbb{H}^3$



(b) Coxeter diagram

Growth functions of Coxeter systems

$\ell_{(W,S)} : W \to \mathbb{N} \cup \{0\}$ the length function
$A_n^{(W,S)} = \{g \in W \mid \ell_{(W,S)} = n\}$
$a_n^{(W,S)} = |A_n^{(W,S)}|$
$f_S(t) = \sum_{n \in \mathbb{N} \cup \{0\}} a_n^{(W,S)} t^n \in \mathbb{Z}[[t]]$ the growth function
The growth rate of (W, S)

$$\begin{aligned}\tau_{(W,S)} &= \limsup_{n\to\infty} \sqrt[n]{a_n^{(W,S)}} \\ &= 1/R_S \ \ (R_S : \text{the radius of convergence of } \ f_S(t)).\end{aligned}$$

Theorem (Solomon 66)

The growth function $f_S(t)$ of an irreducible finite Coxeter group (W, S) can be written as $f_S(t) = \prod_{i=1}^{k}[m_i + 1]$ where $[n] := 1 + t + \cdots + t^{n-1}$ and $\{m_1, m_2, \cdots, m_k\}$ is the set of exponents of (W, S).

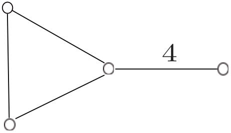
Group	Exponents	Growth function
A_n	$1, 2, \cdots, n-1, n$	$[2, 3, \cdots, n, n+1]$
B_n	$1, 3, \cdots, 2n-3, 2n-1$	$[2, 4, \cdots, 2n-2, 2n]$
D_n	$1, 3, \cdots, 2n-3, n-1$	$[2, 4, \cdots, 2n-2][n]$
$G_2^{(m)}$	$1, m-1$	$[2, m]$
F_4	$1, 5, 7, 11$	$[2, 6, 8, 12]$
H_3	$1, 5, 9$	$[2, 6, 10]$
H_4	$1, 11, 19, 29$	$[2, 12, 20, 30]$

Theorem (Steinberg 68)

Let (W_T, T) be the Coxeter subgroup of (W, S) generated by $T \subseteq S$, and let its growth function be $f_T(t)$. Set $\mathcal{F} = \{\, T \subseteq S \ : \ W_T \text{ is finite} \,\}$. Then

$$\frac{1}{f_S(t^{-1})} = \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_T(t)}.$$

That is, the growth function of (W, S) is a rational function.

Example


type of subgroup	growth function	number
B_3	$[2,4,6]$	2
$A_2 \times A_1$	$[2,2,3]$	1
A_2	$[2,3]$	3
B_2	$[2,4]$	1
$A_1 \times A_1$	$[2,2]$	2
A_1	$[2]$	4

Theorem (Steinberg 68)

Let (W_T, T) be the Coxeter subgroup of (W, S) generated by $T \subseteq S$, and let its growth function be $f_T(t)$. Set $\mathcal{F} = \{ T \subseteq S \ : \ W_T \text{ is finite} \}$. Then

$$\frac{1}{f_S(t^{-1})} = \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_T(t)}.$$

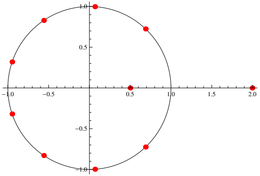
$$\frac{1}{f_S(t^{-1})} = \frac{-2}{[2,4,6]} + \frac{-1}{[2,2,3]} + \frac{3}{[2,3]} + \frac{1}{[2,4]} + \frac{2}{[2,2]} + \frac{-4}{[2]} + 1.$$

$$\begin{aligned} f_S(t) &= \frac{(1+t)^3(1+t^2)(1-t+t^2)(1+t+t^2)}{(t-1)(t^7+t^6+2t^5+2t^4+t^3+t^2-1)} \\ &= 1 + 4t + 10t^2 + 21t^3 + 40t^4 + 73t^5 + \cdots \end{aligned}$$

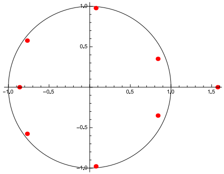
$\frac{1}{f_S(t^{-1})} = \tilde{Q}(t)/\tilde{P}(t) \Rightarrow f_S(t) = P(t)/Q(t)$
where $P(t) = t^n \tilde{P}(1/t),\ Q(t) = t^n \tilde{Q}(1/t)$.
Hence $R = 1/\tau$ is the smallest positive root of $Q(t)$.
Since $\tilde{Q}(t)$ is monic, $\tau > 1$ is an algebraic integer.
$P(t)$ is a product of cyclotomic polynomials.

For compact Coxeter polyhedron P,
$f_S(t) = P(t)/Q(t)$ is reciprocal (*i.e.* $f_S(t^{-1}) = f_S(t)$) when *dim* P is even, while
$f_S(t)$ is antireciprocal (*i.e.* $f_S(t^{-1}) = -f_S(t)$) when *dim* P is odd
(Serre 71).

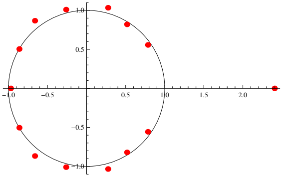
A real algebraic integer $\tau > 1$ is called:
(1) a *Salem number* if τ^{-1} is a conjugate of τ and all conjugates of τ other than τ and τ^{-1} lie on the unit circle.
(2) a *Pisot number* if all algebraic conjugates of τ other than τ lie in the open unit disk.
(3) a *Perron number* if all of whose conjugates have strictly smaller absolute values. (i.e. Salem, Pisot $\Rightarrow$ Perron)



(c) Salem



(d) Pisot



(e) Perron

Growth rates for low dimensional cases

Theorem (Cannon-Wagreich 92, Parry 93)

The growth rates of cocompact 2 and 3-dimensional hyperbolic Coxeter groups are Salem numbers. In practice, Parry showed that the denominator polynomial $Q(x)$ of the growth function of cocompact 2 and 3-dimensional hyperbolic Coxeter group is a product of distinct irreducible cyclotomic polynomials with exactly one Salem polynomial.

Theorem (Floyd 92)

The growth rates of cofinite 2-dimensional hyperbolic Coxeter groups are Pisot numbers.

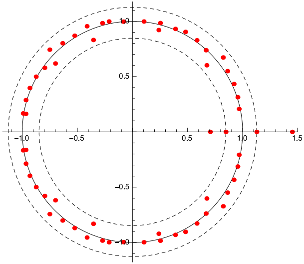
Theorem (Yukita 18)

The growth rates of cofinite 3-dimensional hyperbolic Coxeter groups are Perron numbers.

Conjectures (Kellerhals and Perren 11)

Let (W, S) be a Coxeter group acting on $\mathbb{H}^n$ cocompactly.

1. For n even, $f_S(t)$ has precisely $\frac{n}{2}$ poles $0 < t_1 < \cdots < t_{\frac{n}{2}} < 1$ in the interval $(0, 1)$.
2. For n odd, $f_S(t)$ has a pole at 1 and precisely $\frac{n-1}{2}$ poles $0 < t_1 < \cdots < t_{\frac{n-1}{2}} < 1$ in the interval $(0, 1)$.

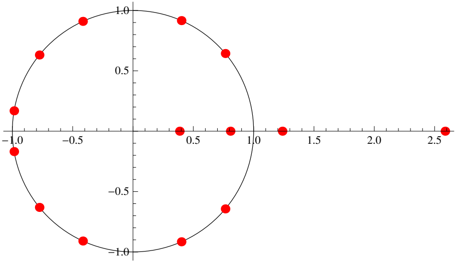


In both cases, all poles are simple, and the non-real poles are contained in the annulus of radius t_i for some $i \in \{1, \cdots, [\frac{n}{2}]\}$.

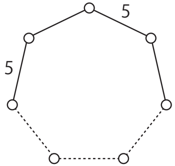
2-Salem numbers as growth rates of 4-dimensional groups

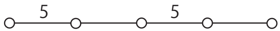
Definition (Samet 52, Kerada 95)

A 2-Salem number is a real algebraic integer $\alpha > 1$, such that α has one conjugate root $\beta > 1$ while other conjugate roots ω satisfy $|\omega| \leq 1$ and at least one of them is on the unit circle. Call the minimal polynomial of α a 2-Salem polynomial. As in the case of a Salem polynomial, a 2-Salem polynomial is a palindromic polynomial of even degree. As a consequence, α^{-1} and β^{-1} are also roots and all roots different from $\alpha, \alpha^{-1}, \beta, \beta^{-1}$ lie on the unit circle.



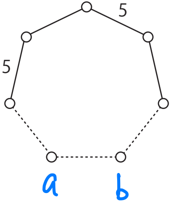
Construction of a truncated 4-dimensional simplex

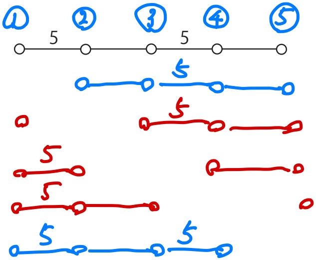


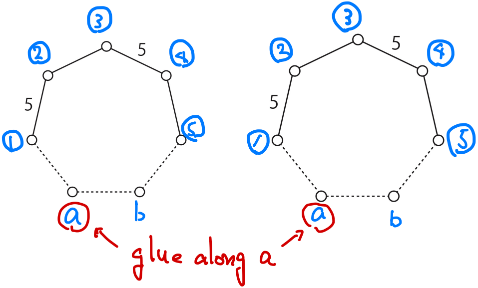


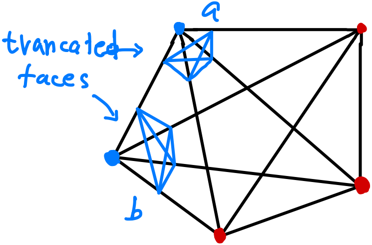
$$G(-\cos\theta_{ij}) = \begin{pmatrix} 1 & -\cos\pi/5 & 0 & 0 & 0 \\ -\cos\pi/5 & 1 & -1/2 & 0 & 0 \\ 0 & -1/2 & 1 & -\cos\pi/5 & 0 \\ 0 & 0 & \cos\pi/5 & 1 & -1/2 \\ 0 & 0 & 0 & -1/2 & 1 \end{pmatrix}$$

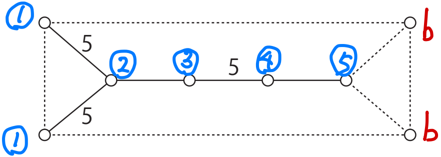
$detG < 0 \Rightarrow$ *the signature of* $G = (4,1) \Rightarrow$ *realizable in* H^4


5
5
a
b

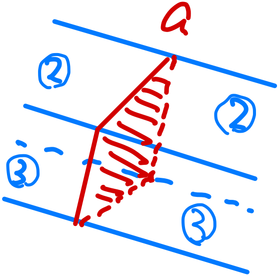

1
2
3
4
5
5
5

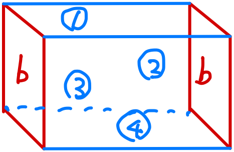

glue along a


a
truncated faces
b



(f) glued Coxeter diagram





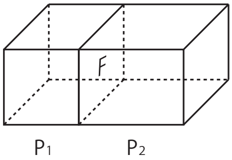
G

b is perpendicular to 1, 2, 3 and 4.

Gluing formula (T. Zehrt and C. Zehrt 11)

Consider two Coxeter n-polytope P_1 and P_2 having the same orthogonal facet F which is a Coxeter (n-1)-polytope, and let their growth functions be $W_1(t), W_2(t)$ and $F(t)$ respectively. Then the growth function $W_1 *_{P_0} W_2(t)$ of the Coxeter polytope obtained by gluing P_1 and P_2 along F is given by

$$\frac{1}{W_1 *_F W_2(t)} = \frac{1}{W_1(t)} + \frac{1}{W_2(t)} + \left(\frac{t-1}{1+t}\right)\frac{1}{F(t)}$$



(g) gluing aong F

Coxeter garlands (T. Zehrt and C. Zehrt 11)

Let G_n be the Coxeter polytope constructed from n copies of G by (n-1)-gluings along orthogonal facets of G. Then the growth function of G_n is equal to $[2, 2, 5, 6](t^5 + 1)/D_n(t)$ where

$$\begin{aligned} D_n(t) &= t^{16} - 2(n+1)t^{15} + t^{14} + (n-1)t^{13} + t^{12} + nt^{11} \\ &+(n-1)t^{10} + 2t^9 + 2(n-1)t^8 + 2t^7 + (n-1)t^6 \\ &+nt^5 + t^4 + (n-1)t^3 + t^2 - 2(n+1)t + 1. \end{aligned}$$

They showed that $D_n(t)$ has 2 reciprocal pairs of positive real zeros and all the other zeros locate on the unit circle. Hence Coxeter garlands seem to have 2-Salem numbers as their growth rates. But it might be possible that it factorizes as a product of cyclotomic polynomials and Salem polynomials...

Theorem 2.1

For any $n \in \mathbb{N}$, $D_n(t)$ is a 2-Salem polynomial. Therefore 4-dimensional Coxeter garlands always have 2-Salem numbers as their growth rates.

Idea of the proof

The Mahler measure of $f(z) = \sum_{0\le k\le d} a_k x^k = a_d \prod_{k=1}^{d}(z - z_k)$ is

$$M(f) = |a_d| \prod_{1\le k\le d} \max\{1, |z_k|\}.$$

Then

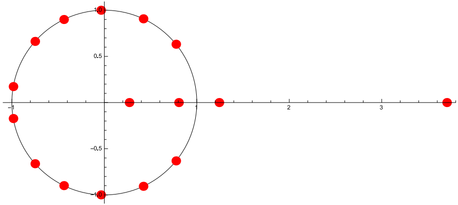
$$|a_k| \le \binom{d}{k} M(f).$$

Hence for fixed $d \in \mathbb{N}$ and fixed $M > 0$, there are only finitely many polynomials $f(z) \in \mathbb{Z}[z]$ with $deg(f) \le d$ and $M(f) \le M$.

T. Zehrt and C. Zehrt 11: Theorem 2

The real roots $1 < \alpha_n < \beta_n$ of $D_n(t)$ of degree 16 satisfy

$$1 < \alpha_n < 2 < 2n + 1 < \beta_n < 2n + 2.$$



In practice Boyd and Mossingohoff classified $\mathbb{Z}$-polynomials with $deg(f) \leq 16$ and $M(f) < 2$ all of which do not factorize $D_n(t)$. Therefore $D_n(t)$ is irreducible over $\mathbb{Z}$.

Lists of Polynomials with Small Mahler Measure (Mossinghoff)

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```
6   1.40126836793985   1-z^2-z^3-z^4+z^6
6   1.50613567955384   1-z-z^3-z^5+z^6
6   1.55603019132268   1-z-z^2+z^3-z^4-z^5+z^6
6   1.58234718368046   1-z^2-2*z^3-z^4+z^6
6   1.63557312992222   1-2*z+2*z^2-3*z^3+2*z^4-2*z^5+z^6
6   1.78164359860800   1-z-z^2-z^4-z^5+z^6
6   1.83107582510231   1-2*z+z^3-2*z^5+z^6
6   1.94685626827188   1-z-z^2-z^3-z^4-z^5+z^6
6   1.96355303898882   1-2*z-z^2+3*z^3-z^4-2*z^5+z^6
6   1.97481870829771   1-2*z+z^2-2*z^3+z^4-2*z^5+z^6
6   1.98779316684622   1-2*z^2-3*z^3-2*z^4+z^6
6   2.04249053394081   1-3*z+3*z^2-3*z^3+3*z^4-3*z^5+z^6
6   2.19564673649222   1-z-z^2-3*z^3-z^4-z^5+z^6
6   2.20113035058076   1-2*z-z^2+2*z^3-z^4-2*z^5+z^6
6   2.22586791398338   1-3*z+2*z^2-z^3+2*z^4-3*z^5+z^6
6   2.25645514685613   1-2*z-z^3-2*z^5+z^6
6   2.26844468522511   1-z-2*z^2-z^3-2*z^4-z^5+z^6
6   2.38214865205606   1-4*z+6*z^2-7*z^3+6*z^4-4*z^5+z^6
6   2.39629574140999   1-2*z-2*z^3-2*z^5+z^6
6   2.42122995630770   1-3*z+z^2+z^3+z^4-3*z^5+z^6
6   2.45317002237950   1-3*z+2*z^2-2*z^3+2*z^4-3*z^5+z^6
6   2.45963365088960   1-z-2*z^2-3*z^3-2*z^4-z^5+z^6
6   2.47541210131401   1-3*z+3*z^2-5*z^3+3*z^4-3*z^5+z^6
6   2.50382260970148   1-2*z-z^2-z^4-2*z^5+z^6
6   2.51454252585576   1-2*z-3*z^3-2*z^5+z^6
6   2.54204886300850   1-z-2*z^2-4*z^3-2*z^4-z^5+z^6
6   2.67509946332357   1-4*z^2-7*z^3-4*z^4+z^6
6   2.69311767304055   1-z-3*z^2-3*z^3-3*z^4-z^5+z^6
6   2.71824589309279   1-2*z-z^2-2*z^3-z^4-2*z^5+z^6
6   2.72706946010527   1-2*z-2*z^2+z^3-2*z^4-2*z^5+z^6
6   2.75146476230721   1-3*z+2*z^2-4*z^3+2*z^4-3*z^5+z^6
6   2.76295394334256   1-z-3*z^2-4*z^3-3*z^4-z^5+z^6
6   2.76701935053725   1-3*z+z^2-z^3+z^4-3*z^5+z^6
6   2.78618956786788   1-3*z+2*z^3-3*z^5+z^6
```