NON-ARITHMETIC LATTICES

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Symmetric spaces

A Riemannian manifold M is a symmetric space if for every point $p \in M$ the map $-Id : T_p(M) \longrightarrow T_p(M)$
extends to a global isometry of M.

Examples

- spaces of constant curvature: $\mathbb{R}^n$, S^n, $\mathbf{H}^n$.
- G semisimple Lie group with maximal compact subgroup K and Riemannian, then $X = G/K$ is a symmetric space.

A symmetric space is of non-compact type if it has non-positive sectional curvatures.

The rank of a symmetric space M is the dimension of the largest Euclidean space that may be totally geodesically, isometrically locally embedded into M.

For example, a geodesic is *locally* an isometric embedding of $\mathbb{R}$ and so all symmetric spaces have rank at least one.

Hyperbolic spaces

The hyperbolic spaces are the rank 1 symmetric spaces of non-compact type. They are

- Real hyperbolic n space $\mathbf{H}^n_{\mathbb{R}}$ for $n \geq 1$;
  $G = \mathrm{SO}^0(n,1)$ and $K = \mathrm{SO}(n)$.
- Complex hyperbolic n space $\mathbf{H}^n_{\mathbb{C}}$ for $n \geq 2$;
  $G = \mathrm{SU}(n,1)$ and $K = \mathrm{S}\big(\mathrm{U}(n) \times \mathrm{U}(1)\big) \simeq \mathrm{U}(n)$.
- Quaternionic hyperbolic n space $\mathbf{H}^n_{\mathbb{H}}$ for $n \geq 2$;
  $G = \mathrm{Sp}(n,1)$ and $K = \mathrm{Sp}(n) \times \mathrm{Sp}(1)$.
- The octonionic hyperbolic plane $\mathbf{H}^2_{\mathbb{O}}$.
  $G = F_{4(-20)}$ and $K = \mathrm{Spin}(9)$.

In fact $\mathbf{H}^1_{\mathbb{C}} \simeq \mathbf{H}^2_{\mathbb{R}}$, $\mathbf{H}^1_{\mathbb{H}} \simeq \mathbf{H}^4_{\mathbb{R}}$, $\mathbf{H}^1_{\mathbb{O}} \simeq \mathbf{H}^8_{\mathbb{R}}$ hence above values of n.
For example $\mathrm{SU}(1,1)$ conjugate to $\mathrm{SL}(2,\mathbb{R})$.

Lattices

Let G be a locally compact topological group with Haar measure. A discrete subgroup Γ of G is a lattice in G if the quotient space $\Gamma \backslash G$ has finite volume.

A lattice Γ is uniform if $\Gamma \backslash G$ is compact and it is non-uniform otherwise.

Suppose G semisimple Lie group with associated symmetric space $X = G/K$ where K is a maximal compact and Riemannian metric g. Then

- Γ acts properly discontinuously on X,
- the quotient space $\Gamma \backslash X$ has finite volume.

Examples

- $\mathbb{Z}^n < \mathbb{R}^n$ with quotient a n-dimensional flat torus T^n.
- The modular group $\mathrm{PSL}(2, \mathbb{Z})$ – see later slides.

The first example is a uniform lattice, the second is non-uniform.

Example – triangle groups

Consider a triangle with sides L_1, L_2, L_3 and internal angles π/a, π/b, π/c where $a, b, c \in \mathbb{N} \cup \{\infty\}$ (where $\pi/\infty = 0$)
Triangle is hyperbolic (resp Euclidean, spherical)
$1/a + 1/b + 1/c < 1$ (resp $= 1, > 1$).

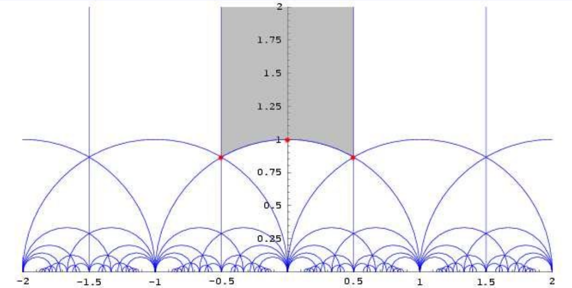
Let R_1, R_2, R_3 be reflection in L_1, L_2, L_3.
The group $\Delta_{a,b,c}$ generated by these reflections is a lattice in $\mathrm{Isom}(\mathbf{H}^2)$ (resp $\mathrm{Isom}(\mathbb{R}^2)$, $\mathrm{Isom}(S^2)$) with presentation:
$\langle R_1, R_2, R_3 \,|\, R_1^2 = R_2^2 = R_3^2 = (R_2R_3)^a = (R_3R_1)^b = (R_1R_2)^c = I \rangle$

Sometimes we consider the index 2 orientation preserving group
$\Gamma_{a,b,c} = \langle A, B \,|\, A^a = B^b = (AB)^c = I \rangle$
Here $A = R_2R_3$, $B = R_3R_1$ so $AB = R_2R_3R_3R_1 = R_2R_1$.
There is a coset decomposition $\Delta_{a,b,c} = \Gamma_{a,b,c} \cup R_3\Gamma_{a,b,c}$.

Example – the modular group $\mathrm{PSL}(2,\mathbb{Z}) = \Gamma_{2,3,\infty}$

$\mathrm{SL}(2,\mathbb{Z})$ generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
It is discrete because $\mathbb{Z}$ is discrete in $\mathbb{R}$.

These act on $\mathbf{H}^2$ by the Möbius transformations in $\mathrm{PSL}(2,\mathbb{Z})$
$S : z \longmapsto -1/z, \quad T : z \longmapsto z+1.$
Fundamental domain a triangle with angles $0, \pi/3, \pi/3$.



Commensurability

Lattices Γ_1 and Γ_2 in G are commensurable if there exists $A \in G$ so that $\Gamma_1 \cap (A\Gamma_2 A^{-1})$ has finite index in both Γ_1 and $A\Gamma_2 A^{-1}$.

For $n \geq 3$ define the Hecke group $H_n = \Gamma_{2,n,\infty}$ generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T_n = \begin{pmatrix} 1 & 2\cos(\pi/n) \\ 0 & 1 \end{pmatrix}$.

- H_3 is the modular group. Note $2\cos(\pi/3) = 1$.
- Consider H_4. Since $2\cos(\pi/4) = \sqrt{2}$ can't immediately conclude discreteness from matrix entries.
  However, let $A = \begin{pmatrix} 2^{1/4} & 0 \\ 0 & 2^{-1/4} \end{pmatrix}$.
  Then $H_3 \cap (AH_4A^{-1})$ generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.
  This has index 3 in H_3 and index 2 in AH_4A^{-1}.
  So H_3, H_4 commensurable, hence H_4 discrete.
- On the other hand, H_5 is not commensurable to H_3 (or to H_4)

Arithmeticity

A linear algebraic group defined over $\mathbb{Q}$ is

- $\mathcal{G} \subset \mathrm{GL}(m, \mathbb{C})$
- coefficients satisfy a set of polynomial equations with coefficients in $\mathbb{Q}$.

Let $\mathcal{G}_{\mathbb{Z}} = \mathcal{G} \cap \mathrm{GL}(m, \mathbb{Z})$, $\mathcal{G}_{\mathbb{R}} = \mathcal{G} \cap \mathrm{GL}(m, \mathbb{R})$.
Then $\mathcal{G}_{\mathbb{Z}}$ is an arithmetic subgroup of $\mathcal{G}_{\mathbb{R}}$.

Let G be a semisimple Lie group.
Let $\phi : \mathcal{G}_{\mathbb{R}} \longrightarrow G$ be a continuous, surjective homomorphism with compact kernel.

Then $\Gamma < G$ is arithmetic if Γ is commensurable with $\phi(\mathcal{G}_{\mathbb{Z}})$.
(i.e.there exists $A \in G$ so that $A\Gamma A^{-1} \cap \phi(\mathcal{G}_{\mathbb{Z}})$ has finite index in both $A\Gamma A^{-1}$ and $\phi(\mathcal{G}_{\mathbb{Z}})$.)
Example.

- $(\mathrm{P})\mathrm{SL}(2, \mathbb{Z})$ – direct from definition
- the Hecke group H_4 – using commensurability to $\mathrm{SL}(2, \mathbb{Z})$.

An example of an arithmetic group

Let Q be the quadratic form $Q = \mathbf{diag}(1, \ldots, 1, -\sqrt{2})$.
Let $\mathrm{SO}(Q)$ be the group of unimodular real matrices preserving Q.
Then Q has signature $(n, 1)$ and $\mathrm{SO}(Q)$ is isomorphic to $\mathrm{SO}(n, 1)$.

Let $\Gamma = \mathrm{SO}(Q) \cap \mathrm{SL}(n+1, \mathbb{Z}[\sqrt{2}])$ be the group of unimodular matrices with entries in $\mathbb{Z}[\sqrt{2}]$ preserving Q.
As $\mathbb{Z}[\sqrt{2}]$ is not discrete in $\mathbb{R}$ we cannot deduce that Γ is discrete.

Let σ be the Galois automorphism of $\mathbb{Q}(\sqrt{2})$ sending $\sqrt{2}$ to $-\sqrt{2}$.
Let $Q^\sigma = \mathbf{diag}(1, \ldots, 1, \sqrt{2})$. This has signature $(n+1, 0)$.

If $A \in \Gamma$ let A^σ be matrix obtained by applying σ to entries of A.
Then $A^\sigma \in \mathrm{SO}(Q^\sigma) \cap \mathrm{SL}(n+1, \mathbb{Z}[\sqrt{2}])$.

$\widehat{\Gamma} = \{(A, A^\sigma) \mid A \in \Gamma\}$ is a discrete subgroup of $\mathrm{SO}(Q) \times \mathrm{SO}(Q^\sigma)$.
Let $\phi : \mathrm{SO}(Q) \times \mathrm{SO}(Q^\sigma) \longrightarrow \mathrm{SO}(Q)$ be map onto the first factor.
Then $\ker(\phi) = \mathrm{SO}(Q^\sigma)$ which is compact.

Therefore $\Gamma = \phi(\widehat{\Gamma})$ is discrete. It is an arithmetic group.

Takeuchi's theorem

Consider the orientation preserving subgroups $\Gamma_{a,b,c}$ of the triangle groups $\Delta_{a,b,c}$ where $a, b, c \in \mathbb{N} \cup \{\infty\}$ with $1/a + 1/b + 1/c < 1$.
Note that $\Gamma_{a,b,c}$ is a lattice in $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{Isom}_0(\mathbf{H}^2)$.

Theorem (Takeuchi 1977)
There are only finitely many triples (a, b, c) for which $\Gamma_{a,b,c}$ is arithmetic. For all other triples the group $\Gamma_{a,b,c}$ is non-arithmetic. Moreover, there are infinitely many commensurability classes of non-arithmetic triangle groups.

The only Hecke groups that are arithmetic are H_3, H_4, H_6.
In particular, H_5 is non-arithmetic – so not commensurable to H_3.

Takeuchi's list

THEOREM 3. *The complete list of all triples* (e_1, e_2, e_3) *of arithmetic type is as follows:*

(i) *Compact types.*

(2, 3, 7), (2, 3, 8), (2, 3, 9), (2, 3, 10), (2, 3, 11), (2, 3, 12), (2, 3, 14), (2, 3, 16), (2, 3, 18), (2, 3, 24), (2, 3, 30), (2, 4, 5), (2, 4, 6), (2, 4, 7), (2, 4, 8), (2, 4, 10), (2, 4, 12), (2, 4, 18), (2, 5, 5), (2, 5, 6), (2, 5, 8), (2, 5, 10), (2, 5, 20), (2, 5, 30), (2, 6, 6), (2, 6, 8), (2, 6, 12), (2, 7, 7), (2, 7, 14), (2, 8, 8), (2, 8, 16), (2, 9, 18),

(2, 10, 10), (2, 12, 12), (2, 12, 24), (2, 15, 30), (2, 18, 18),

(3, 3, 4), (3, 3, 5), (3, 3, 6), (3, 3, 7), (3, 3, 8), (3, 3, 9), (3, 3, 12), (3, 3, 15), (3, 4, 4), (3, 4, 6), (3, 4, 12), (3, 5, 5), (3, 6, 6), (3, 6, 18), (3, 8, 8), (3, 8, 24), (3, 10, 30), (3, 12, 12),

(4, 4, 4), (4, 4, 5), (4, 4, 6), (4, 4, 9), (4, 5, 5), (4, 6, 6), (4, 8, 8), (4, 16, 16),

(5, 5, 5), (5, 5, 10), (5, 5, 15), (5, 10, 10),

(6, 6, 6), (6, 12, 12), (6, 24, 24), (7, 7, 7), (8, 8, 8), (9, 9, 9), (9, 18, 18), (12, 12, 12), (15, 15, 15).

(ii) *Non-compact types.*

$(2, 3, \infty)$, $(2, 4, \infty)$, $(2, 6, \infty)$, $(2, \infty, \infty)$, $(3, 3, \infty)$, $(3, \infty, \infty)$, $(4, 4, \infty)$, $(6, 6, \infty)$, (∞, ∞, ∞).

Lattices and arithmeticity

In irreducible symmetric spaces X of non-compact type:

- All arithmetic groups are lattices (Borel & Harish-Chandra).
- All lattices are arithmetic when:
  - $\text{Rank}(X) > 1$ (Margulis).
  - X is either $\mathbf{H}^n_{\mathbb{H}}$ or $\mathbf{H}^2_{\mathbb{O}}$ (Corlette, Gromov & Schoen).
- There exist non-arithmetic lattices in $\mathrm{SO}(n,1)$ (so $X = \mathbf{H}^n_{\mathbb{R}}$) for all n (Gromov & Piatetski-Shapiro).

This only leaves $\mathrm{SU}(n,1)$ (so the case of $X = \mathbf{H}^n_{\mathbb{C}}$).

- There exist non-arithmetic lattices in $\mathrm{SU}(2,1)$ (Mostow 1980, Deligne-Mostow 1986, Deraux-P-Paupert 2016, 2021). Currently 22 commensurability classes known.
- There exist non-arithmetic lattices in $\mathrm{SU}(3,1)$ (Deligne-Mostow 1986, Deraux 2020). Currently 2 commensurability classes known.

Open problem:
Do there exist non-arithmetic lattices in $\mathrm{SU}(n,1)$ for $n \geq 4$?

Descriptions of the lattices in $\mathrm{SU}(2,1)$

Four ways to describe complex hyperbolic lattices:

- Using arithmeticity – not good when lattices non-arithmetic.
- Using hyperbolic geometry to build fundamental domains; Mostow (1980), Deraux-P-Paupert (2016,2021)
- Using algebraic geometry – they are ball quotients whose Chern classes satisfy $c_1^2 = 3c_2$ Yau (1968), Miyaoka (1983).
  - For Deligne-Mostow lattices , Hirzebruch (1983, 1984), Shvartsman (1992).
  - For some Deraux-P-Paupert lattices Deraux (2018, 2019).
- As monodromy groups on certain moduli spaces.
  - Using hypergeometric functions in two variables (order 2 differential equation) Deligne-Mostow (1986) for the groups they construct.
  - Using higher hypergeometric functions in one variable (order 3 differential equation) P (2021) for all the above groups.

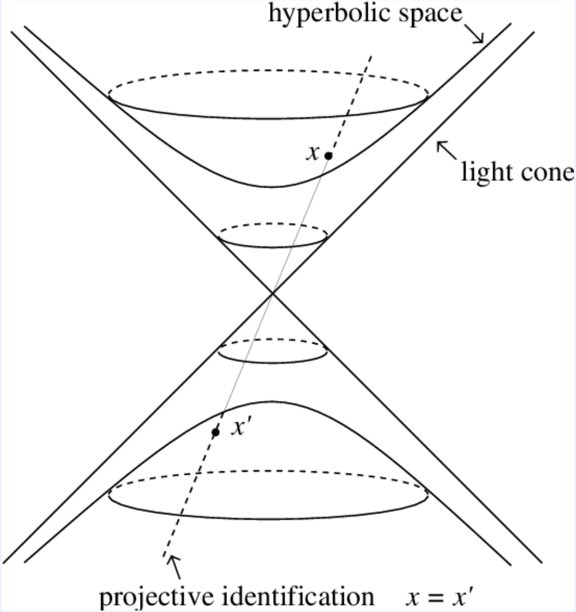
Complex hyperbolic space and its isometries

Let $\mathbb{C}^{n,1}$ be complex vector space with
Hermitian form $H = \langle \cdot\, , \, \cdot \rangle$ of signature $(n, 1)$. Let
$V_- = \left\{ \mathbf{z} \in \mathbb{C}^{n,1} \;:\; \langle \mathbf{z}, \mathbf{z} \rangle < 0 \right\}$.
and $\mathbb{P} : \mathbb{C}^{n,1} - \{\mathbf{0}\} \to \mathbb{CP}^n$ be canonical projection.
Complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}} = \mathbb{P}V_-$.

A useful model is unit ball in $\mathbb{C}^n$:
Take $\langle \mathbf{z}, \mathbf{z} \rangle = |z_1|^2 + \cdots + |z_n|^2 - |z_{n+1}|^2$.
If $\mathbf{z} \in V_-$ then $z_{n+1} \neq 0$.
So $(z_1/z_{n+1}, \ldots, z_n/z_{n+1})$ inhomogeneous coordinates on $\mathbf{H}^n_{\mathbb{C}}$.
Finally $\langle \mathbf{z}, \mathbf{z} \rangle < 0$ implies $|z_1/z_{n+1}|^2 + \cdots + |z_n/z_{n+1}|^2 < 1$.

$SU(H)$ group of unimodular matrices preserving H
maximal compact subgroup $K \simeq \mathrm{U}(n)$.
$\mathbf{H}^n_{\mathbb{C}} = SU(H)/K$.
$PSU(H) = SU(H)/\{\lambda I\}$ holomorphic isometry group of $\mathbf{H}^n_{\mathbb{C}}$.

Complex version of Minkowski space



Non-arithmeticity

Non-arithmeticity criterion of Mostow for $SU(H)$
Based on earlier criterion of Vinberg for Coxeter groups..

Theorem (Mostow)

Let E be a totally real number field and F be a purely imaginary quadratic extension of E and $\mathcal{O}_F$ be the ring of integers of F. Let H be a Hermitian form of signature (n,1) defined over F.

- *Suppose $\Gamma \subset SU(H; \mathcal{O}_F)$ is a lattice. Then Γ is arithmetic if and only if for alll $\varphi \in \mathrm{Gal}(F)$ not inducing the identity on E, the form ${}^{\varphi}H$ is definite.*
- *$SU(H; \mathcal{O}_F)$ is a lattice if and only if it is arithmetic.*

To show non-arithmeticity of a lattice $\Gamma \subset SU(H; \mathcal{O}_F)$ it is sufficient to find one φ with indefinite ${}^{\varphi}H$.

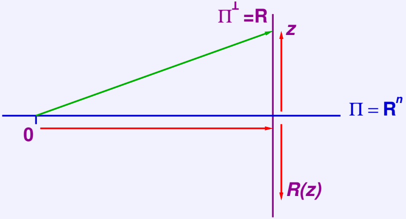
Constructing examples

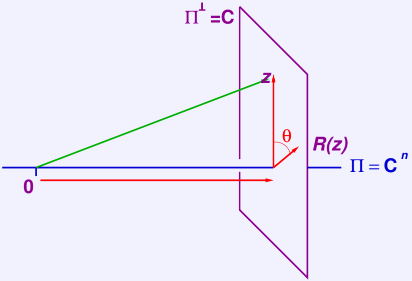
For the rest of the talk we will describe the groups considered by Mostow, Deligne-Mostow and Deraux-P-Paupert.

These will generalise triangle groups $\Delta_{a,b,c}$.

Real and complex reflections

A real reflection $R(\mathbf{z})$ with mirror a real hyperplane Π in $\mathbb{R}^{n+1}$.





A complex reflection $R(\mathbf{z})$ with angle θ and mirror a complex hyperplane Π in $\mathbb{C}^{n+1}$.

Complex reflections in $\mathbb{C}^{n,1}$

Π complex hyperplane through origin in $\mathbb{C}^n$
$\mathbf{n}$ normal vector in $\mathbb{C}^{n,1}$ to Π with respect to H

Complex reflection $R(\mathbf{z})$ with mirror Π and angle θ

$$\begin{aligned} R(\mathbf{z}) &= \left(\mathbf{z} - \frac{\langle \mathbf{z}, \mathbf{n}\rangle}{\langle \mathbf{n}, \mathbf{n}\rangle}\mathbf{n}\right) + (e^{i\theta})\frac{\langle \mathbf{z}, \mathbf{n}\rangle}{\langle \mathbf{n}, \mathbf{n}\rangle}\mathbf{n} \\ &= \mathbf{z} + (e^{i\theta} - 1)\frac{\langle \mathbf{z}, \mathbf{n}\rangle}{\langle \mathbf{n}, \mathbf{n}\rangle}\mathbf{n}. \end{aligned}$$

Complex reflections can have arbitrary order.

Represented by matrix in $U(H)$ with
one eigenvalue $e^{i\theta}$ and n eigenvalues 1.

Multiply by $e^{-i\theta/(n+1)}$ to get matrix in $SU(H)$.

We will be interested in complex hyperbolic 2-space $\mathbf{H}^2_{\mathbb{C}}$.

Braiding

If Π_1 and Π_2 are real hyperplanes that intersect with angle ϕ and R_1 and R_2 are (real) reflections with mirrors Π_1 and Π_2, then $R_1 R_2$ is a rotation through angle 2ϕ fixing $\Pi_1 \cap \Pi_2$.
In particular, if $\phi = \pi/n$ then $(R_1 R_2)^n = I$.

For complex reflections this notion is replaced by braiding.
R_1 and R_2 satisfy a braid relation of length n, $\mathrm{br}_n(R_1, R_2)$ if

- $(R_1 R_2)^m = (R_2 R_1)^m$ if $n = 2m$ is even;
- $(R_1 R_2)^m R_1 = (R_2 R_1)^m R_2$ if $n = 2m + 1$ is odd.

Note:

1. $\mathrm{br}_2(R_1, R_2)$ says R_1 and R_2 commute: $R_1 R_2 = R_2 R_1$;
2. $\mathrm{br}_3(R_1, R_2)$ is the classical braid relation: $R_1 R_2 R_1 = R_2 R_1 R_2$.
3. If R_1 and R_2 both have order 2 then
   $\mathrm{br}_n(R_1, R_2)$ if and only if $(R_1 R_2)^n = I$.

Complex hyperbolic triangle groups

Recall the real triangle groups $\Delta_{a,b,c}$ rom the start of the talk
$\langle R_1, R_2, R_3 \mid R_1^2 = R_2^2 = R_3^2 = (R_2R_3)^a = (R_1R_3)^b = (R_1R_2)^c = I \rangle$

- Consider three complex lines L_1 L_2, L_3 in $\mathbf{H}^2_{\mathbb{C}}$.
- Let R_1, R_2, R_3 be complex reflections in these lines, each with angle $2\pi/p$. Therefore
  the relations R_1^2, R_2^2, R_3^2 are replaced with R_1^p, R_2^p, R_3^p.
- At each vertex the power of the product is replaced with a braid relation:
  the relation $(R_2R_3)^a$ is replaced with $\mathrm{br}_a(R_2, R_3)$;
  the relation $(R_1R_3)^b$ is replaced with $\mathrm{br}_b(R_1, R_3)$;
  the relation $(R_1R_2)^c$ is replaced with $\mathrm{br}_c(R_1, R_2)$.
- To determine the triangle (and the group) we need an extra parameter. We suppose $\mathrm{br}_d(R_1, R_3^{-1}R_2R_3)$.
- There will be further relations.....

The groups we consider

The group is a subgroup of $\mathrm{SU}(2,1)$
generated by three complex reflections $R_1,\ R_2,\ R_3$
Each of $R_1,\ R_2,\ R_3$ has rotation angle $2\pi/p$, so order $p \geq 2$.
They satisfy the following braid relations.
$\mathrm{br}_a(R_2, R_3)$, $\mathrm{br}_b(R_1, R_3)$, $\mathrm{br}_c(R_1, R_2)$, $\mathrm{br}_d(R_1, R_3^{-1}R_2R_3)$.

In what follows we use the following conventions:

- $(a, b, c; d)$ gives a an allowable set of braid relations;
- For each $(a, b, c; d)$ we list the allowable orders of reflection p;
- If p is red then the group is non-arithmetic;
  if p is blue then the group is arithmetic.

The lattices with $a = b = c = 3$ were constructed by
Mostow (1980), Livné (1981), Deligne-Mostow (1986).
The others were constructed by Deraux-P-Paupert (2016, 2021).
The lattices with $b \neq c$ follow from ideas of Thompson.

a	*b*	*c*	*d*	*p*
3	3	3	2	5, 6, 7, 8, 9, 10, 12, 18
3	3	3	3	4, 5, 6, 7, 8, 9, 10, 12, 18
3	3	3	4	3, 4, 5, 6,8, 12
3	3	3	5	3, 4, 5, 10
3	3	3	6	3, 4, 6
3	3	3	7	3, 7
3	3	3	8	3, 4
3	3	3	9	3
3	3	3	10	3
3	3	3	12	3
4	4	4	3	3, 4, 5, 6, 8, 12
4	4	4	4	3, 4, 5, 6, 8, 12
4	4	4	5	2, 3, 4
5	5	5	3	3, 4, 5, 10
5	5	5	5	2, 3, 4, 5, 10
6	6	6	4	3, 4, 6

a	b	c	d	p
3	3	4	4	3, 4, 5, 6, 8, 12
3	3	5	5	2, 3, 5, 10
4	4	3	3	4, 5, 6, 8, 12
5	5	4	4	3, 4, 5
3	3	4	3	3, 4, 5, 6, 8, 12
3	3	4	5	3, 4, 5
3	3	4	6	3, 4, 5
3	3	4	7	2, 7
2	3	3	3	5, 6, 7, 8, 9, 10, 12, 18
2	3	4	4	4, 5, 6, 8, 12
2	3	5	5	3, 4, 5, 10
2	3	6	6	3, 4, 6
3	4	4	4	3, 4, 6, 12

Sample presentations

$(a, b, c; d) = (6, 6, 6; 4)$

$$\left\langle R_1, R_2, R_3, J : \begin{array}{l} R_2 = JR_1J^{-1},\ R_3 = J^{-1}R_1J = JR_2J^{-1}, \\ R_1^p,\ J^3,\ (R_1J)^8,\ (R_1R_2)^{\frac{3p}{p-3}},\ (R_1R_3^{-1}R_2R_3)^{\frac{4p}{p-4}}, \\ \mathrm{br}_6(R_1, R_2),\ \mathrm{br}_4(R_1, R_3^{-1}R_2R_3) \end{array} \right\rangle.$$

$(a, b, c; d) = (4, 4, 4; 3)$

$$\left\langle R_1, R_2, R_3, J : \begin{array}{l} R_2 = JR_1J^{-1},\ R_3 = J^{-1}R_1J = JR_2J^{-1}, \\ R_1^p,\ J^3,\ (R_1J)^7,\ (R_1R_2)^{\frac{4p}{p-4}},\ (R_1R_3^{-1}R_2R_3)^{\frac{6p}{p-6}}, \\ \mathrm{br}_4(R_1, R_2),\ \mathrm{br}_3(R_1, R_3^{-1}R_2R_3) \end{array} \right\rangle.$$

$(a, b, c; d) = (3, 3, 4; 5)$

$$\left\langle R_1, R_2, R_3 : \begin{array}{l} R_1^p,\ (R_1R_2R_3)^5,\ (R_1R_2)^{\frac{4p}{p-4}},\ (R_1R_3^{-1}R_2R_3)^{\frac{10p}{3p-10}}, \\ \mathrm{br}_3(R_2, R_3),\ \mathrm{br}_3(R_3, R_1),\ \mathrm{br}_4(R_1, R_2), \\ \mathrm{br}_5(R_1, R_3^{-1}R_2R_3) \end{array} \right\rangle.$$

THANK YOU FOR YOUR ATTENTION!