Automorphism groups of extremal Klein surfaces

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Purpose

We present some automorphism groups of extremal Klein surfaces.

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Klein surface: a topological surface together with a dianalytic structure
Every coordinate transition function of the surface is analytic or anti-analytic.

In this talk a Klein surface means a non-orientable closed ones.

Extremal surfaces

S: a closed hyperbolic surface of genus g.

$$g = \begin{cases} \#\text{handles} \geqq 2 & \text{if } S \text{ is orientable,} \\ \#\text{cross caps} \geqq 3 & \text{if } S \text{ is non-orientable.} \end{cases}$$

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$\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$

$\exists \Gamma$: non-Euclidean crystallographic group (NEC group) s.t.

$$S = \mathbb{H}/\Gamma.$$

NEC group: a discrete cocompact subgroup of

$$\operatorname{Aut}^{\pm}(\mathbb{H}) = \mathrm{PGL}(2, \mathbb{R})$$

S inherits the hyperbolic metric from $\mathbb{H}$.

$S = \mathbb{H}/\Gamma$: an extremal surface

$\overset{\text{def}}{\Longleftrightarrow}$ S contains an extremal disc D, i.e. the largest disc of which radius r_g is determined by g and the orientability of S.

$$\cosh r_g = \frac{1}{2\sin(\pi/N)},$$

where $N = \begin{cases} 12g - 6 & \text{(if } S \text{ is orientable)} \\ 6g - 6 & \text{(if } S \text{ is non-orientable)} \end{cases}$

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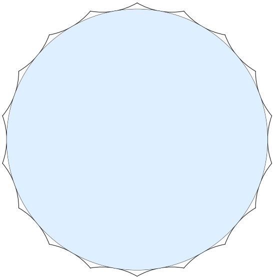
$\overset{\text{iff}}{\Longleftrightarrow}$ $\exists$ a regular N-gon as a fundamental region of Γ,

$\overset{\text{iff}}{\Longleftrightarrow}$ Γ is torsion-free subgroup of

$\begin{cases} \Delta(2,3,N) \text{ with index } N & \text{(if } S \text{ is orientable)} \\ \Delta^{\pm}(2,3,N) \text{ with index } 2N & \text{(if } S \text{ is non-orientable)} \end{cases}$

How large is an extremal disc?



Closed Riemann surface of genus 2

$$\frac{\text{area}(\boldsymbol{D})}{\text{area}(\boldsymbol{S})} \approx \mathbf{0.9397.}$$

(This hyperbolic polygon is depicted in the unit disc $\mathbb{D}$.)

The limit of the ration as $g \to \infty$

S_g: an extremal Riemann surface of genus $g \geqq 2$
D_g: an extremal disc in S_g
Then

$$\frac{\text{area}(D_g)}{\text{area}(S_g)} = \frac{2\pi(\cosh r_g - 1)}{2\pi(2g-2)} = \frac{\frac{1}{2\sin(\pi/(12g-6))} - 1}{2g-2}$$

$$\xrightarrow{g\to\infty} \frac{3}{\pi} \approx 0.9549.$$

Extremal Klein surfaces

Extremal surfaces may contain more than one extremal discs. However ...

Theorem 1.1 (Girondo-N (2007))

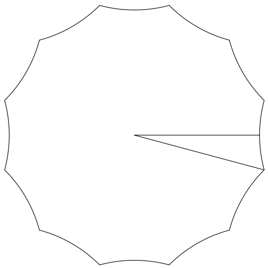
If $g > 6$, then non-orientable extremal surfaces of genus g do not contain more than one extremal disc.

Theorem 1.2 (Girondo-N (2007))

S is extremal iff $\exists\Gamma$: a torsion-free NEC subgroup of $\Delta^{\pm}(2,3,6g-6)$ with index $12g-12$ s.t. $S=\mathbb{H}/\Gamma$

$\Delta^{\pm}(p,q,r)$: the extended triangle group
i.e. the group generated by three reflections of sides of a triangle with angle $\pi/p, \pi/q, \pi/r$.

Proof. (⇒) S has a regular $6g-6$-gon P as its fundamental region.
$\Gamma := \langle$side-pairings of $P\rangle$
Every side-pairing of P is generated by $\sigma_1, \sigma_2, \sigma_3$ (reflections).



Therefore $\Gamma \subset \Delta^{\pm} = \Delta^{\pm}(2, 3, 6g-6)$.

$$[\Delta^{\pm} : \Gamma] = \frac{\text{Area}(\Gamma)}{\text{Area}(\Delta^{\pm})} = 12g - 12.$$

(⇐) Since Γ is an NEC group, S is non-orientable. S is extremal because it has a regular $6g-6$-gon as the fundamental region.

Commensurators

G: a Fuchsian group

$$\mathrm{Comm}(G) := \left\{ \begin{array}{l} \alpha \in \mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}(2,\mathbb{R}); \\ G \cap \alpha^{-1}G\alpha \text{ has finite index} \\ \text{in } G \text{ and in } \alpha^{-1}G\alpha \end{array} \right\}$$

G: an NEC group

$$\mathrm{Comm}^{\pm}(G) := \left\{ \begin{array}{l} \alpha \in \mathrm{Aut}^{\pm}(\mathbb{H}) = \mathrm{PGL}(2,\mathbb{R}); \\ G \cap \alpha^{-1}G\alpha \text{ has finite index} \\ \text{in } G \text{ and in } \alpha^{-1}G\alpha \end{array} \right\}$$

Theorem 1.3

G: an NEC group
$G^+ := G \cap \mathbf{Aut}(\mathbb{H})$ is nonarithmetic
$\Rightarrow \mathbf{Comm}^{\pm}(G)$ is discrete

Let G be a (proper) NEC group. Since $[G : G^+] = 2$, $G = G^+ \cup \gamma G^+$ for some $\gamma \in G$.

Lemma 1.4

$$\mathrm{Comm}^{\pm}(G^+) = \mathrm{Comm}(G^+) \cup \gamma\ \mathrm{Comm}(G^+)$$

Lemma 1.5

$$\mathrm{Comm}^{\pm}(G^+) = \mathrm{Comm}^{\pm}(G)$$

Proof of Thm 1.3

G^+ is nonarithmetic

$\Rightarrow$ Comm(G^+) is discrete (Margulis' theorem).

$\Rightarrow$ Comm$^{\pm}(G^+)$ is discrete Lem 1.4.

$\Rightarrow$ Comm$^{\pm}(G)$ is discrete Lem 1.5.

Uniqueness of discs

Theorem 1.6 (Girondo-N (2007))

S: a non-orientable extremal surface of genus $g > 6$
$\Rightarrow$ S has a unique extremal disc.

Proof. D_1, D_2: extremal discs of $S = \mathbb{H}/K$
For each D_i, $\exists \Delta_i^{\pm}(2, 3, 6g-6) \supset K$
$\exists \alpha \in \mathrm{Aut}^{\pm}(\mathbb{H})$ s.t. $\alpha^{-1}\Delta_1^{\pm}\alpha = \Delta_2^{\pm}$
Then $\alpha \in \mathrm{Comm}^{\pm}(\Delta_1^{\pm})$.

$\mathbf{\Delta}_1^+ := \mathbf{\Delta}_1^\pm \cap \mathrm{Aut}(\mathbb{H})$, triangle group of type $(2, 3, 6g-6)$

$$\left(\begin{array}{l} \underline{\text{K.Takeuchi } (1977)} \\ \text{List of all arithmetic triangle groups} \\ \text{e.g., } (2,3,n): \text{ arithmetic} \\ \Rightarrow n = 7, 8, 9, 10, 11, 12, 14, 16, 18, 24, 30. \end{array}\right)$$

$g > 6$

$\Rightarrow \mathbf{\Delta}_1^+$ is nonarithmetic

$\Rightarrow \mathrm{Comm}^\pm(\mathbf{\Delta}_1^\pm)$ is discrete (Thm 2).

$$\left(\frac{\text{E. Bujalance } (1982)}{\text{normal pairs of NEC signatures}} \quad \frac{\text{J.L.Estévez and M.Izquierdo } (2006)}{\text{non-normal pairs of NEC signatures}}\right)$$

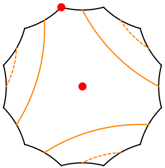
$\Rightarrow \boldsymbol{\Delta_1^{\pm}}$ is maximal.
$\Rightarrow \mathrm{Comm}^{\pm}(\boldsymbol{\Delta_1^{\pm}}) = \boldsymbol{\Delta_1^{\pm}}$
$\Rightarrow \boldsymbol{\alpha} \in \boldsymbol{\Delta_1^{\pm}}$
$\Rightarrow \boldsymbol{\Delta_1^{\pm}} = \boldsymbol{\Delta_2^{\pm}}$
$\Rightarrow \boldsymbol{D_1} = \boldsymbol{D_2}$

Theorem 1.7 (Girondo-N (2007))

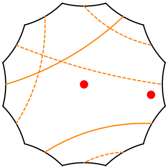
There exist exactly 11 non-orientable extremal surfaces $S_1, \ldots, S_{11}$ *of genus 3. All the centers of extremal discs are obtained. The group of automorphisms of these surfaces are also obtained.*

The next slide shows the fundamental regions of NEC groups for these surfaces.

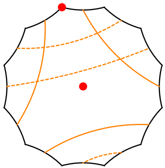
S_1



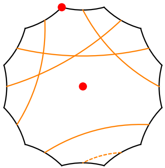
S_2



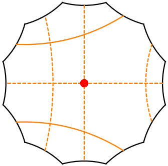
S_3



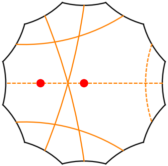
S_4



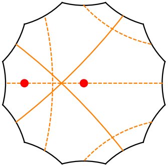
S_5



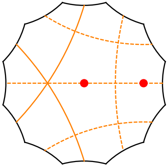
S_6



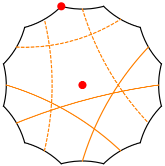
S_7



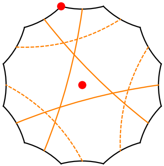
S_8



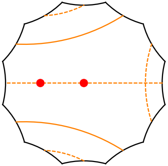
S_9



S_{10}



S_{11}



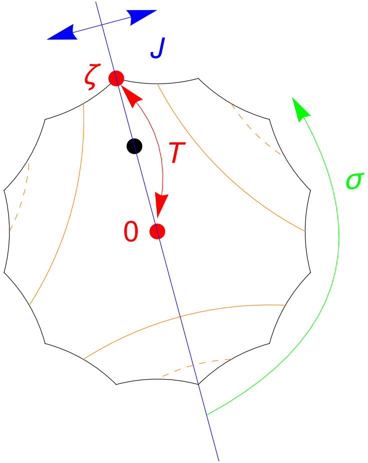
The group of automorphisms of S_1

$\mathrm{Aut}(S_1) = D_3 \times \mathbb{Z}_2 = \langle \sigma, J \rangle \times \langle JT \rangle$

$t(z) = \dfrac{\zeta - z}{1 - \bar{\zeta} z} \rightsquigarrow T,$

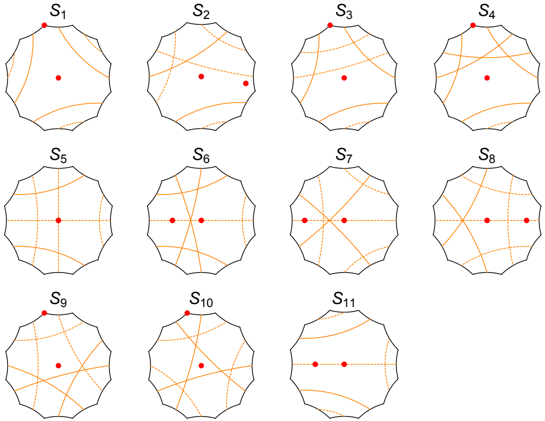
$s(z) = e^{2\pi i/3} z \rightsquigarrow \sigma.$

$j(z) = e^{-5\pi i/6} \bar{z} \rightsquigarrow J$



The group of automorphisms of $S_1, \ldots, S_{11}$

Surface	Aut
S_1	$D_3 \times \mathbb{Z}_2$
S_2	$\mathbb{Z}_2$
S_3	$\mathbb{Z}_2 \times \mathbb{Z}_2$
S_4	$\mathbb{Z}_2 \times \mathbb{Z}_2$
S_5	$\mathbb{Z}_2$
S_6	$\mathbb{Z}_2 \times \mathbb{Z}_2$
S_7	$\mathbb{Z}_2 \times \mathbb{Z}_2$
S_8	$\mathbb{Z}_2 \times \mathbb{Z}_2$
S_9	$\mathbb{Z}_2 \times \mathbb{Z}_2$
S_{10}	$D_3 \times \mathbb{Z}_2$
S_{11}	$\mathbb{Z}_2 \times \mathbb{Z}_2$



For $g = 4, 5, 6$, the groups of automorphisms are obtained [N2009], [N2012], [N2016].

The maximum order of automorphisms

Theorem 2.1 (W. Hall, 1978)

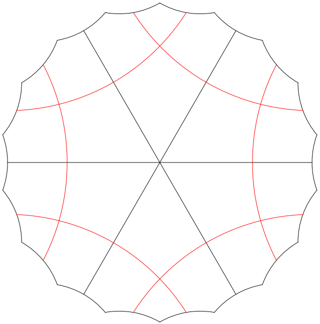
The maximum order of an automorphism of a closed non-orientable surface of genus $g \geqq 3$ is

$$\begin{cases} 2g & \text{if } g \text{ is odd,} \\ 2(g-1) & \text{if } g \text{ is even.} \end{cases}$$

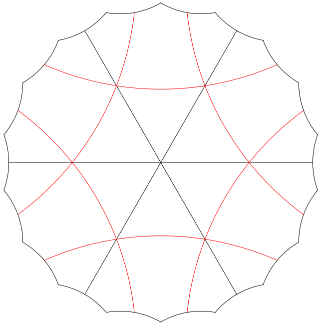
In the next slides black (resp. red) curves indicate pairings of sides by orientation-preserving (resp. reversing) side-paring mappings.

Corollary 2.2 (N)

There exist exactly two non-orientable extremal surfaces of genus 4 which admits the maximum order $\mathbf{6 = 2(g-1)}$.



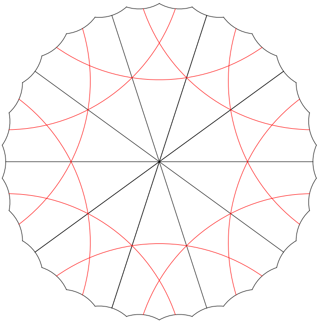
$\boldsymbol{X}_4$



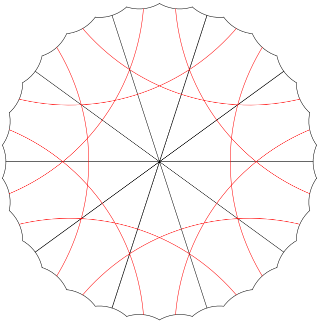
$\boldsymbol{Y}_4$

Corollary 2.3 (N)

There exist exactly two non-orientable extremal surfaces of genus 6 which admits the maximum order $10 = 2(g-1)$.



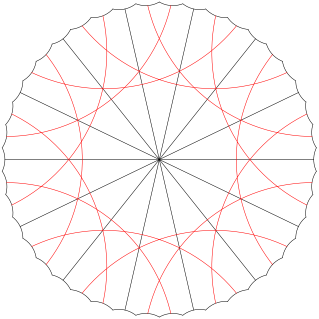
X_6



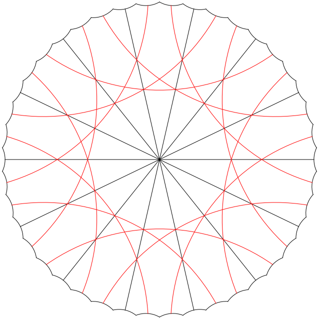
Y_6

Corollary 2.4 (N)

There exist exactly two non-orientable extremal surfaces of genus 8 which admits the maximum order $\mathbf{14 = 2(g-1)}$.



$\boldsymbol{X_8}$



$\boldsymbol{Y_8}$

The side-pairing patterns

Number the sides of 30-gon from 0 to 29. Then the side-pairing patterns of $\boldsymbol{X_6}$ and $\boldsymbol{Y_6}$ are as follows.

$\boldsymbol{X_6}$			$\boldsymbol{Y_6}$		
0	—	15	0	—	15
3	—	18	3	—	18
6	—	21	6	—	21
9	—	24	9	—	24
12	—	27	12	—	27
1	—	8	1	—	5
4	—	11	4	—	8
7	—	14	7	—	11
	⋮			⋮	
28	—	5	28	—	2

As an application of Corollaries 2.2-2.4, we can show that there are two families $\{X_{2m}\}_{m=2}^{\infty}$ and $\{Y_{2m}\}_{m=2}^{\infty}$ of non-orientable extremal surfaces of even genus $g = 2m$ with an automorphism of order $2(g-1)$.

Proposition 2.5 (N)

Let $S = \mathbb{D}/\Gamma$ *be a compact non-orientable surface of genus* $g = 2m$ $(m \geq 2)$*. If* S *is a non-orientable extremal surface with an automorphism of maximum order* $2(g-1)$*, then a fundamental region of the NEC group* Γ *is taken as the regular* $6(g-1)$*-gon of which side-paring pattern is either 1 or 2 as follows:*

1. $$\begin{array}{ccll} 3i & — & 3i+6m-3 & (i = 0, 1, \ldots, 2m-2), \\ 3i+1 & — & 3i+3m-1 & (i = 0, 1, \ldots, 4m-3), \end{array}$$
2. $$\begin{array}{ccll} 3i & — & 3i+6m-3 & (i = 0, 1, \ldots, 2m-2), \\ 3i+1 & — & 3i+9m-4 & (i = 0, 1, \ldots, 4m-3). \end{array}$$

Proposition 2.5 (N)

Let $S = \mathbb{D}/\Gamma$ be a compact non-orientable surface of genus $g = 2m$ $(m \geq 2)$. If S is a non-orientable extremal surface with an automorphism of maximum order $2(g-1)$, then a fundamental region of the NEC group Γ is taken as the regular $6(g-1)$-gon of which side-paring pattern is either 1 or 2 as follows:

1. $$\begin{array}{ccll} 3i & - & 3i+6m-3 & (i = 0, 1, \ldots, 2m-2), \\ 3i+1 & - & 3i+3m-1 & (i = 0, 1, \ldots, 4m-3), \end{array}$$
2. $$\begin{array}{ccll} 3i & - & 3i+6m-3 & (i = 0, 1, \ldots, 2m-2), \\ 3i+1 & - & 3i+9m-4 & (i = 0, 1, \ldots, 4m-3). \end{array}$$

Thank you so much!